Linear algebra: determinants and eigenvalues/eigenvectors

Statistical Natural Language Processing 1

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Quick recap

So far we reviewed:

- Vectors, matrices
- Operations on vectors and matrices: scalar multiplication, addition, dot product, matrix multiplication
- Matrices as operators (linear functions / transformations)
- Linearity and linear combinations
- Solving systems of linear equations, elimination
- Finding matrix inverse
- Linear regression

Today's plan

- Determinant
- Eigenvalues and eigenvectors

Determinant

- The determinant of a square matrix is a number that provides a lot of information about the matrix
  - Whether the matrix has an inverse or not
  - Calculating eigenvalues and eigenvectors
  - Solving systems of linear equations
  - Determining the (signed) ‘change of volume’ caused by the linear transformation defined by the matrix

Calculating the determinant

- The determinant of a 2x2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

- The determinant of larger matrices are defined recursively
  - Choose a row or column
  - The determinant is the sum of the each element in the row (or column) multiplied by its *cofactor*
  - The cofactor of an element a_{ij} is the determinant of 'sub-matrix' (or *minor*) multiplied by -1^{i+j}
  - The minor of a_{ij} is the matrix obtained by removing row i and column j from the original matrix

Calculating the determinant

exercise

$$\begin{vmatrix} 2 & 2 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} =$$

Calculating the determinant

exercise

$$\begin{vmatrix} 2 & 2 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2 \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 4 \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

Calculating the determinant

exercise

$$\begin{vmatrix} 2 & 2 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2 \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 4 \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= 2 \times 1 - 2 \times 0 + 4 \times (-1)$$

Calculating the determinant

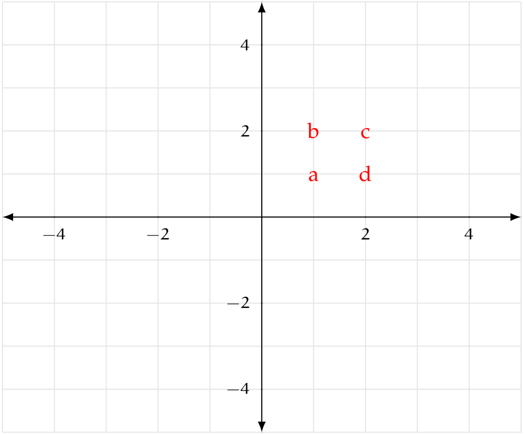
exercise

$$
\begin{aligned}
\begin{vmatrix} 2 & 2 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} &= 2 \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 4 \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\
&= 2 \times 1 - 2 \times 0 + 4 \times (-1) \\
&= -1
\end{aligned}
$$

Determinant

example geometric interpretation (1)

- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- $\det(A) = ?$



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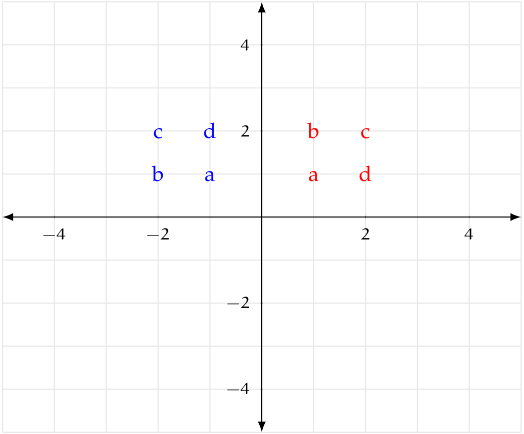
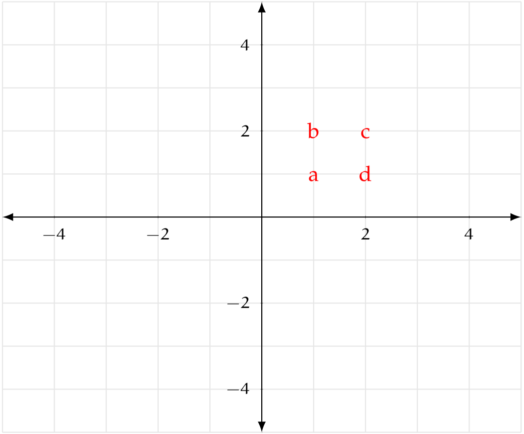


Determinant

example geometric interpretation (2)

- $A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$
- $\det(A) = ?$



Determinant

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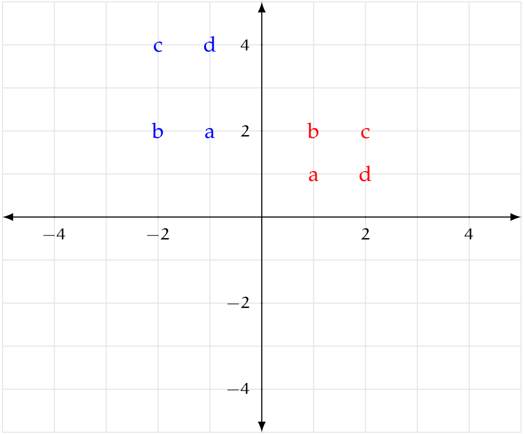
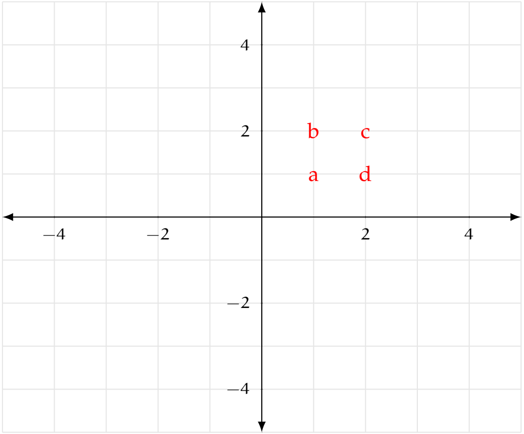


Determinant

example geometric interpretation (3)

- $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$
- $\det(A) = ?$



Determinant

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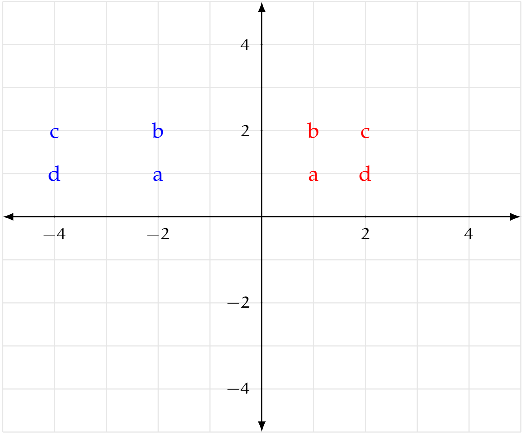
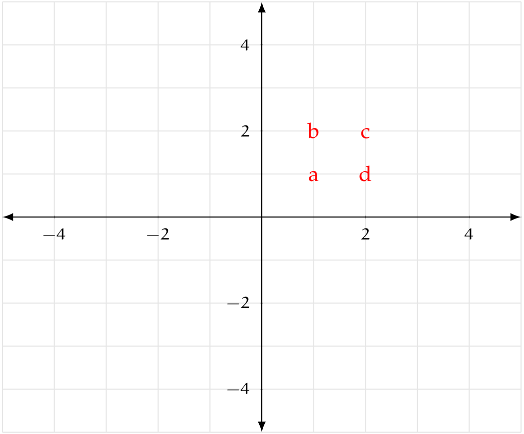


Determinant

example geometric interpretation (3)

- $A = \begin{bmatrix} \cos 120 \\ \sin 120 \end{bmatrix} \times \begin{bmatrix} \cos 120 & \sin 120 \end{bmatrix}$
  $= \begin{bmatrix} 0.25 & -0.43 \\ -0.43 & 0.75 \end{bmatrix}$
- $\det(A) = ?$



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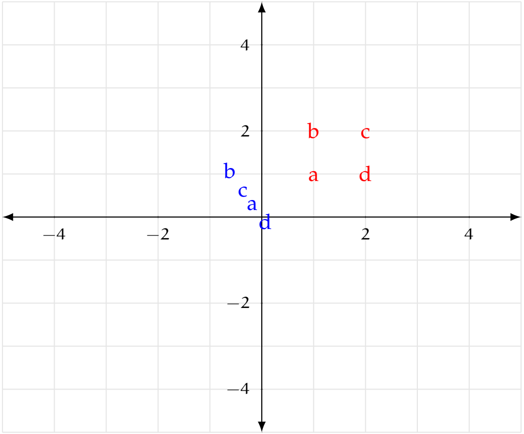


Some properties of determinants

- $\det(\mathbf{I}) = 1$
- The determinant of a triangular matrix is the product of the main diagonal
- If two columns or rows are the same, the determinant is 0
- If we multiply a row of $\boldsymbol{A}$ with a scalar c, determinant becomes $c \times \det A$
- Elementary row operations do not change the determinant (except permutations)
- If we exchange two rows of $\boldsymbol{A}$, determinant becomes $-\det \boldsymbol{A}$
- $\det(\boldsymbol{A}\mathbf{B}) = \det(\boldsymbol{A})\det(\mathbf{B})$
- $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Eigenvalues and eigenvectors

- We can view any linear transformation as a combination of scaling and rotation (and reflection)
- The linear transformation defined by a matrix does not change the directions of some vectors, vectors in these directions are called the *eigenvectors*
- The scaling factor in these directions is called *eigenvalues*
- More formally, if v is an eigenvector of $\boldsymbol{A}$ with corresponding eigenvalue $\boldsymbol{\lambda}$,

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$

- Independent eigenvectors of a symmetric matrix are orthogonal

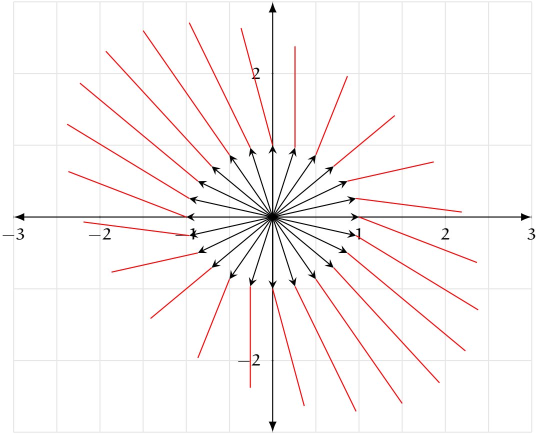
Eigenvalues and eigenvectors

visualization

- We start with the vectors (black arrows)
- The red lines trace the vector after transformation with

$$\begin{bmatrix} 2.3660 & -0.3660 \\ -0.6340 & 2.6340 \end{bmatrix}$$

- In some directions, the vector is only scaled



Finding eigenvalues and eigenvectors

- We can start from the definition

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$

- Rearranging,

$$\boldsymbol{A}\boldsymbol{v} - \lambda\boldsymbol{v} = 0$$

$$(\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{v} = 0$$

- This means the matrix $\boldsymbol{A} - \lambda\mathbf{I}$ should be singular for non-zero $\boldsymbol{v}$, and

$$\det(\boldsymbol{A} - \lambda\mathbf{I}) = 0$$

- Now we can first solve the equation for λ, and knowing λs we can find the corresponding eigenvectors

Finding eigenvalues and eigenvectors

an example

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Finding eigenvalues and eigenvectors

an example

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Solution:

$$\begin{aligned} \lambda_1 &= 5 \\ \lambda_2 &= 3 \\ \boldsymbol{v}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \boldsymbol{v}_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

Properties of eigenvalues and eigenvectors

- An $n \times n$ matrix $\boldsymbol{A}$ has n eigenvalues (which can be complex, or repeated)
- The sum of eigenvalues is the sum of the diagonal of $\boldsymbol{A}$ (the *trace* of $\boldsymbol{A}$)
- The product of the eigenvalues is the determinant
- $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have the same eigenvalues
- For symmetric matrices, the eigenvectors can be chosen to be orthonormal
- If all eigenvalues of a symmetric matrix are positive, it is called a *positive definite* matrix. More formally, if $\boldsymbol{A}$ is positive definite, then $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ is positive for any $\boldsymbol{x}$
- If all eigenvalues of a symmetric matrix are non-negative, it is called a *positive semi-definite* matrix

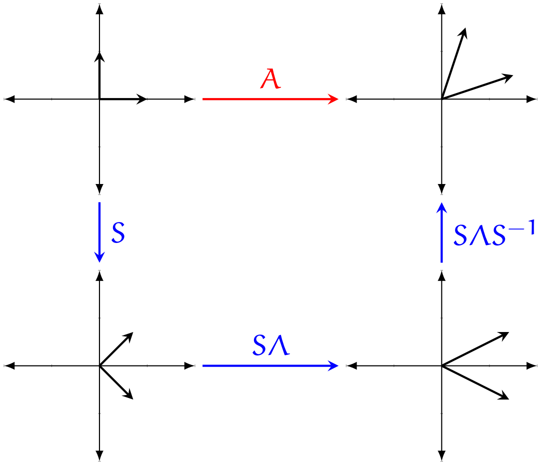
Diagonalization

(eigenvalue decomposition)

- An $n \times n$ with n matrix with distinct eigenvalues can be *diagonalized* using eigenvalues and eigenvectors
- We take the matrix $\mathbf{S}$ whose columns are the eigenvectors of $\mathbf{A}$, and the diagonal matrix $\mathbf{\Lambda}$ with eigenvalues of $\mathbf{A}$, then

$$
\begin{aligned}
\mathbf{AS} &= \mathbf{S\Lambda} \\
\mathbf{A} &= \mathbf{S\Lambda S}^{-1} \\
\mathbf{S}^{-1}\mathbf{AS} &= \mathbf{\Lambda}
\end{aligned}
$$

The geometry of eigenvalue decomposition



Matrix powers and matrix inverse

- Matrix powers can be easily calculated with diagonalization

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\
\mathbf{A}\mathbf{A}\mathbf{x} &= \lambda\mathbf{A}\mathbf{x} \\
\mathbf{A}^2\mathbf{x} &= \lambda^2\mathbf{x}
\end{aligned}
$$

- In general,

$$
\begin{aligned}
\mathbf{A}^2 &= \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1} \\
&= \mathbf{S}\mathbf{\Lambda}^2\mathbf{S}^{-1} \\
\mathbf{A}^k &= \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1}
\end{aligned}
$$

- Inverse is also easy to obtain after eigendecomposition

$$
\mathbf{A}^{-1} = \mathbf{S}\mathbf{\Lambda}^{-1}\mathbf{S}^{-1}
$$

Summary / next

- We reviewed eigenvalues and eigenvectors
- Eigenvalues and eigenvectors have many practical applications from image compression to clustering and dimensionality reduction

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- Eigenvalues and eigenvectors have many practical applications from image compression to clustering and dimensionality reduction

Next:

- SVD and pseudo inverse

Further reading

Any of the linear algebra references provided earlier.