Linear algebra: vectors, matrices, dot product

Statistical Natural Language Processing 1

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Seminar für Sprachwissenschaft

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Some practical remarks

(recap)

- Course web page: `https://snlp1-2025.github.io` (public)
  `https://github.com/snlp1-2025/snlp1/` (private)
- If you haven't done already, please fill in the questionnaire on Moodle

Today's lecture

- Some concepts from linear algebra
  - Vectors
  - Dot product
  - Matrices

This is only a high-level, informal introduction/refresher.

Linear algebra

Linear algebra is the field of mathematics that studies *vectors* and *matrices*.

- A vector is an ordered sequence of numbers

$$\boldsymbol{v} = (6, 17)$$

- A matrix is a rectangular arrangement of numbers

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

- A well-known application of linear algebra is solving a set of linear equations

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 6 \\ x_1 & + & 4x_2 & = & 17 \end{array} \quad \Longleftrightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

Why study linear algebra?

Consider an application counting words in a document

	the	and	of	to	in	...
	121	106	91	83	43	...

Why study linear algebra?

Consider an application counting words in a document

	the	and	of	to	in	...	
(	121	106	91	83	43	...	)

Why study linear algebra?

Consider an application counting words in multiple documents

	the	and	of	to	in	…
document_1	121	106	91	83	43	…
document_2	142	136	86	91	69	…
document_3	107	94	41	47	33	…
…	…	…	…	…	…	

You should already be seeing vectors and matrices here.

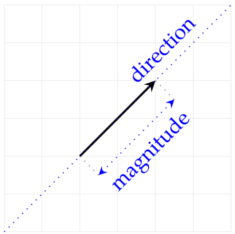
Why study linear algebra?

- Insights from linear algebra are helpful in understanding many NLP methods
- In machine learning, we typically represent input, output, parameters as vectors or matrices (or tensors)
- It makes notation concise and manageable
- In programming, many machine learning libraries make use of vectors and matrices explicitly
- In programming, vector-matrix operations correspond to loops
- 'Vectorized' operations may run much faster on GPUs, and on modern CPUs

Vectors

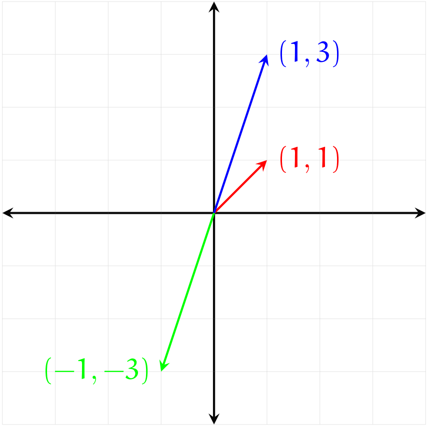
- Vectors are objects with a magnitude and a direction
- We represent vectors with an ordered list of numbers $\boldsymbol{v} = (v_1, v_2, \ldots v_n)$
- The number n (the number of elements or entries of the vector) is its dimension
- We often call an n dimensional vector as n-vector
- The vector of n real numbers is said to be in $\mathbb{R}^n$ ($\boldsymbol{v} \in \mathbb{R}^n$)
- Typical notation for vectors:

$$\boldsymbol{v} = \vec{v} = (v_1, v_2, v_3) = \langle v_1, v_2, v_3 \rangle = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



Geometric interpretation of vectors

- Geometrically, vectors are represented with arrows from the origin in the Euclidean space
- The endpoint of the vector $\boldsymbol{v} = (v_1, v_2)$ correspond to the Cartesian coordinates defined by v_1, v_2
- These generally make sense for two or three-dimensional spaces
- The intuitions often (!) generalize to higher dimensional spaces



Some special vectors

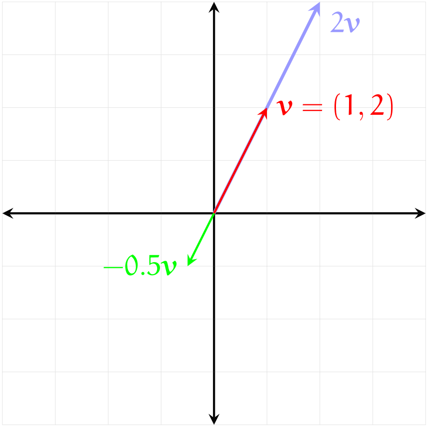
- The zero vector, $\mathbf{0}$, is the vector whose all entries are 0
- The vector of all 1s, $\mathbf{1}$, is also often interesting
- A more interesting set of vectors is *standard unit vectors* (examples below are 4-dimensional standard unit vectors)

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

- n-dimensional standard unit vectors form the standard *basis* for n-dimensional (vector) space
- In some textbooks, standard unit vectors of two (and three) dimensions are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$
- In ML they are related to *one-hot representation*: we represent categorical predictors (variables) with n values as n-dimensional standard unit vectors

Multiplying a vector with a scalar

- For a vector $\boldsymbol{v} = (v_1, v_2)$ and a scalar a,
$$a\boldsymbol{v} = (av_1, av_2)$$
- multiplying with a scalar 'scales' the vector
- We can use the notation $a\mathbf{1}$ for a vector whose all entries are a



Vector addition and subtraction

For vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$

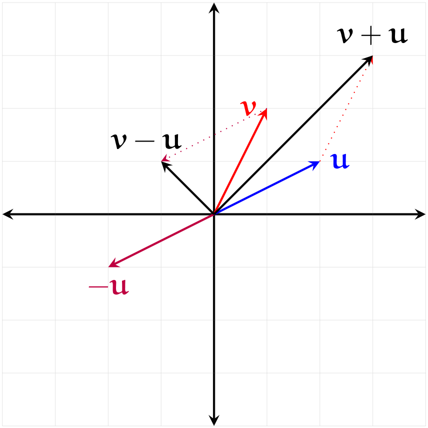
- $\mathbf{v} + \mathbf{u} = (v_1 + u_1, v_2 + u_2)$

$$(1, 2) + (2, 1) = (3, 3)$$

- $\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u})$

$$(1, 2) - (2, 1) = (-1, 1)$$

- For any vector $\mathbf{v}$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$



Properties of vector operations

- Vector addition and scalar multiplication is commutative

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$a\mathbf{u} = \mathbf{u}a$$

- Scalar multiplication and vector addition also show the following distributive properties

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$

Linearity and linear functions

- A linear function $f(\cdot)$ (or mapping) follows
  - $f(a\boldsymbol{v}) = af(\boldsymbol{v})$ (homogeneity)
  - $f(\boldsymbol{v} + \boldsymbol{u}) = f(\boldsymbol{v}) + f(\boldsymbol{u})$ (additivity)
  - combined together: $f(a\boldsymbol{v} + b\boldsymbol{u}) = af(\boldsymbol{v}) + bf(\boldsymbol{u})$
- A combination of vectors as in

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n.$$

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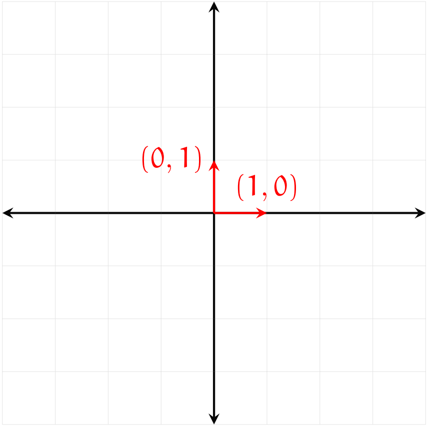
Question: Is $f(x) = ax + b$ linear?

Linear combinations of standard unit vectors

- Any n-vector can be written as a linear combination of standard unit vectors. Example:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- n-dimensional standard unit vectors form a *basis* for $\mathbb{R}^n$
- n-dimensional standard unit vectors form *span* the vector space $\mathbb{R}^n$



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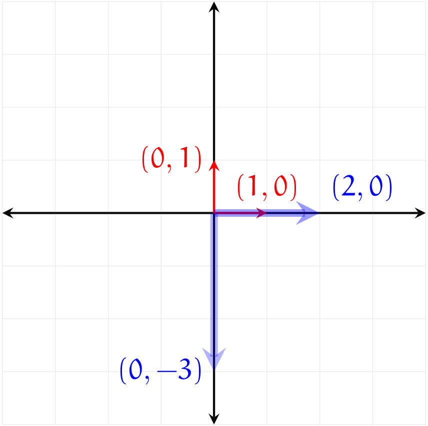

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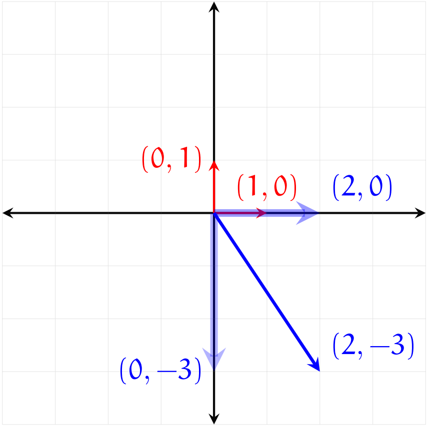

Dot (inner) product

- Dot product is an operation between two vectors with same dimensions

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

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- Calculate the dot products for the following vectors

$$\begin{bmatrix}4\\3\end{bmatrix} \cdot \begin{bmatrix}3\\4\end{bmatrix} \qquad \begin{bmatrix}4\\-3\end{bmatrix} \cdot \begin{bmatrix}-3\\4\end{bmatrix} \qquad \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}-2\\-4\\-6\end{bmatrix} \qquad \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}2\\4\\6\end{bmatrix} \qquad \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

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$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

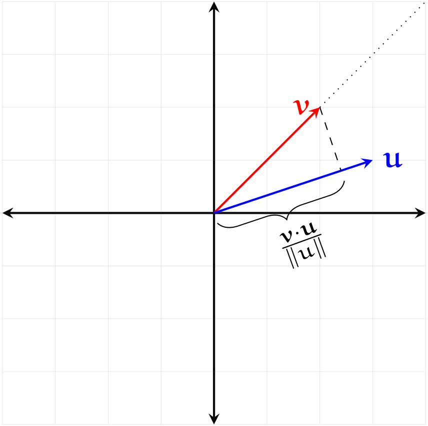
- Note that dot product is larger when the vectors are 'similar'

Properties of dot product

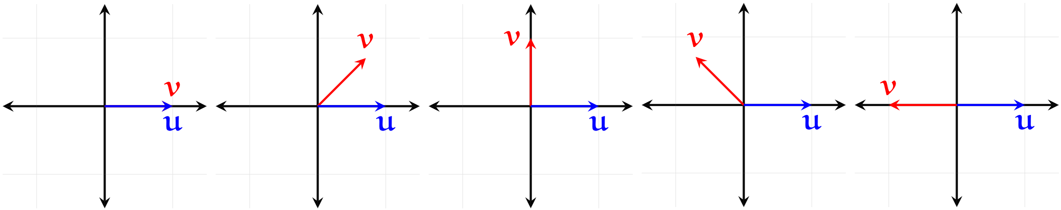
- *Commutativity* $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- *Distributivity with vector addition* $\mathbf{u} \cdot (\mathbf{v} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u}$
- *Associativity with scalar multiplication* $(a\mathbf{u}) \cdot (b\mathbf{v}) = ab(\mathbf{u} \cdot \mathbf{v})$.
- Note that dot product is not associative, since the result of the dot product is not a vector, but a scalar

Geometric interpretation of the dot product

- The dot product of two vectors gives the (orthogonal) projection of one of the vectors to the line defined by the other



Dot product with unit vectors



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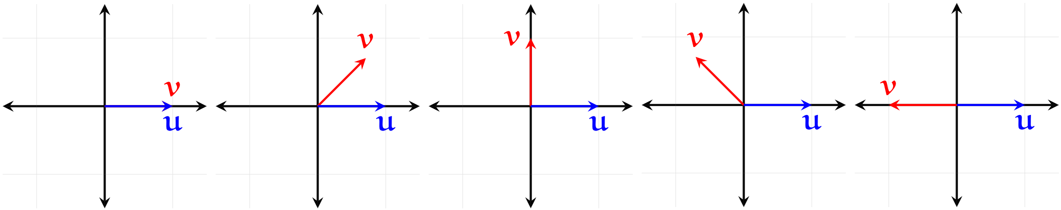
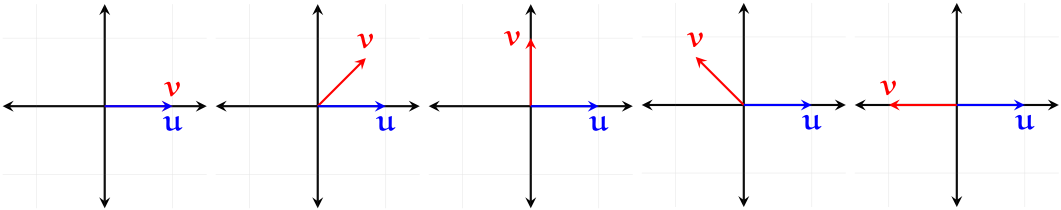


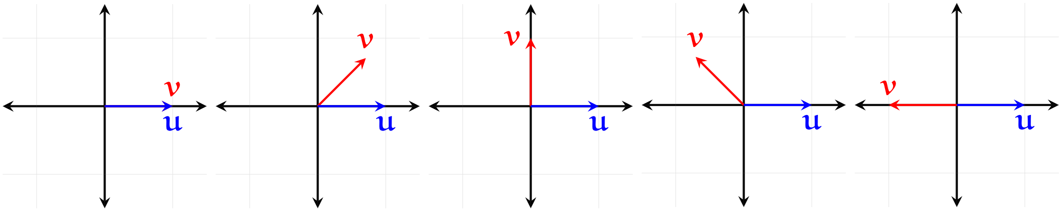
$\mathbf{u} \cdot \mathbf{v} = 1$

Dot product with unit vectors



$\mathbf{u} \cdot \mathbf{v} = 1$ $\mathbf{u} \cdot \mathbf{v} = 0.7$

Dot product with unit vectors



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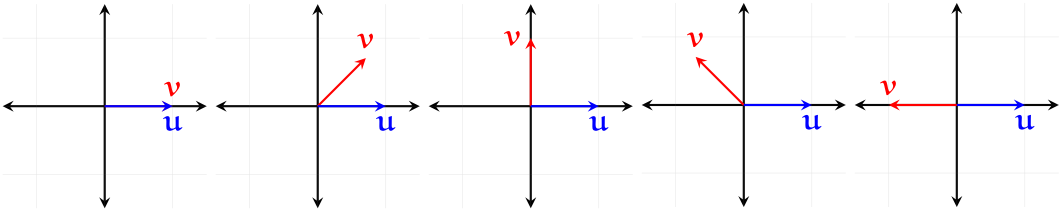
Dot product with unit vectors



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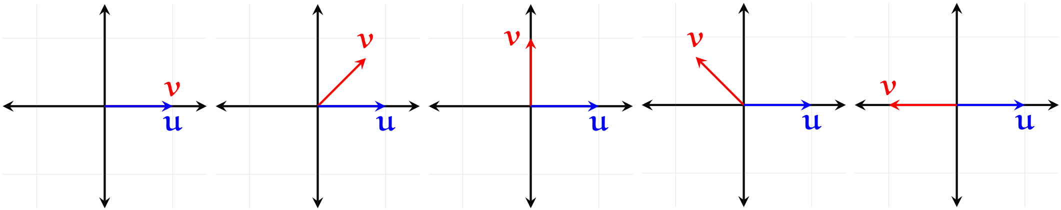
Dot product with unit vectors



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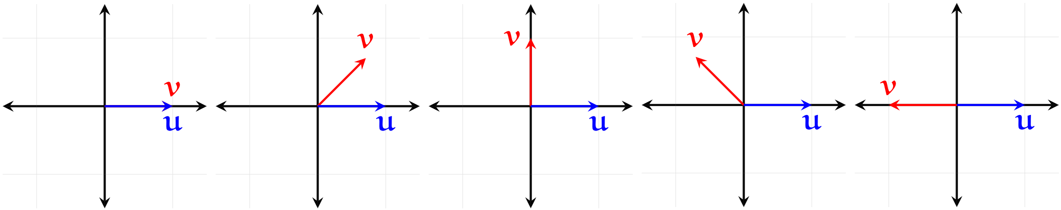
Dot product with unit vectors



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- The dot product is larger if the vectors point to the similar directions
- The dot product is 0 if the vectors are orthogonal

Vector norms

- The *norm* of a vector is an indication of its size (magnitude)
- The norm of a vector is the distance from its tail to its tip
- Norms are related to distance measures
- Vector norms are particularly important for understanding some machine learning techniques

L2 norm

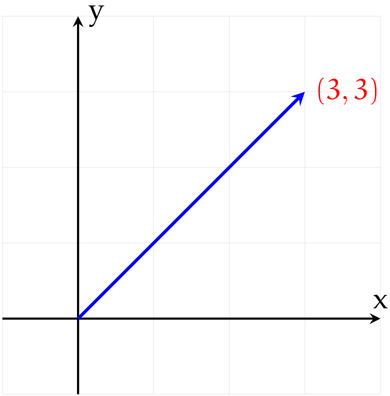
- Euclidean norm, or L2 (or L_2) norm is the most commonly used norm
- For $\boldsymbol{v} = (v_1, v_2, \ldots v_n)$,

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 + \ldots v_n^2}$$
$$= \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$$

- For example,

$$\|(3,3)\|_2 = \sqrt{3^2 + 3^2} = \sqrt{18}$$

- L2 norm is the default, we often skip the subscript $\|\boldsymbol{v}\|$

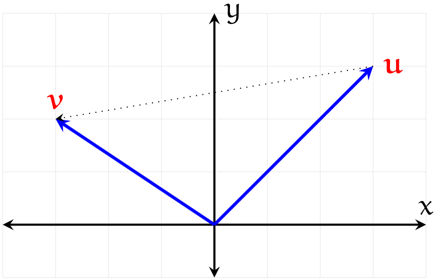


Euclidean distance

- Euclidean distance between two vectors is the L2 norm of their difference

$$D(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-6)^2 + (-1)^2}$$

- Euclidean distance is a metric
  - symmetric $\|\mathbf{v} - \mathbf{u}\| = \|\mathbf{u} - \mathbf{v}\|$
  - non-negative
  - and obeys the triangle inequality $D(\mathbf{u}, \mathbf{v}) \leqslant D(\mathbf{u}, \mathbf{w}) + D(\mathbf{w}, \mathbf{v})$ for any $\mathbf{w}$



Cauchy–Schwarz inequality

$$|\mathbf{u} \cdot \boldsymbol{v}| \leqslant \|u\|\|v\|$$

- In words: the product of the norms of two vectors is greater than or equal to absolute value of their dot product
- Inequality is important as it links norms and the dot product, it is useful proofs of many intersting relations
- Equality holds only when two vectors are aligned

Chebyshev's inequality

Given a vector $\boldsymbol{v} = (v_1, v_2, \ldots v_n)$ with k entries with the property $|x_i| \geqslant a$,

$$\|v\|^2 \geqslant ka^2$$

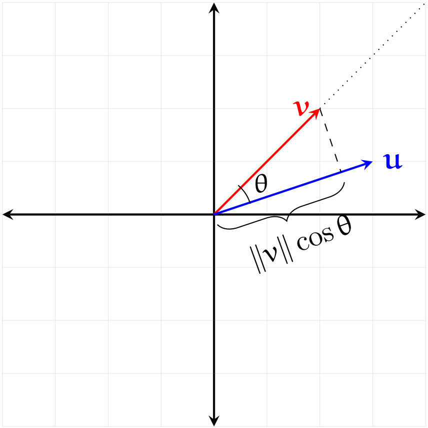
- This puts a limit on the number of large (divergent) values in a vector in terms of its norm
- A trivial consequence: no entry in a vector can be larger than its norm
- The inequality also has important consequences in statistics (we will return to it later)

Cosine similarity

- The cosine of the angle between two vectors

$$\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{u}}{\|\boldsymbol{v}\| \cdot \|\boldsymbol{u}\|}$$

  is called *cosine similarity*
- Unlike dot product, the cosine similarity is not sensitive to the magnitudes of the vectors
- The cosine similarity is bounded in range $[-1, +1]$



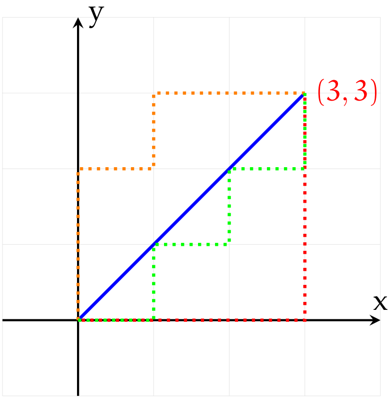
L1 norm

- Another norm we will often encounter is the L1 norm

$$\|v\|_1 = |v_1| + |v_2|$$

$$\|(3,3)\|_1 = |3| + |3| = 6$$

- L1 norm is related to Manhattan distance



L_P norm

In general, L_P norm, is defined as

$$\|\boldsymbol{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}$$

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We will only work with than L1 and L2 norms, but you may also see L_0 and L_∞ norms in related literature

Matrices

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,m} \end{bmatrix}$$

- We can think of matrices as collection of row or column vectors
- A matrix with n rows and m columns is in $\mathbb{R}^{n \times m}$
- Some operations in linear algebra also generalize to more than 2-D objects
- A *tensor* can be thought of a generalization of vectors and matrices to multiple dimensions

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Multiplying a matrix with a scalar

Similar to vectors, each element is multiplied by the scalar.

$$2\begin{bmatrix}2 & 1\\1 & 4\end{bmatrix} = \begin{bmatrix}2\times 2 & 2\times 1\\2\times 1 & 2\times 4\end{bmatrix} = \begin{bmatrix}4 & 2\\2 & 8\end{bmatrix}$$

Matrix addition and subtraction

Each element is added to (or subtracted from) the corresponding element

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

Note:

- Matrix addition and subtraction are defined on matrices of the same dimensions

Transpose of a matrix

Transpose of a $n \times m$ matrix is an $m \times n$ matrix whose rows are the columns of the original matrix.
Transpose of a matrix $\mathbf{A}$ is denoted with $\mathbf{A}^T$.

$$\text{If } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \mathbf{A}^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}.$$

Some special matrices

Identity matrix

- A square matrix in which all the elements of the principal diagonal are one and all other elements are zero is called *identity matrix* ($\mathbf{I}$)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Multiplying a vector and matrix with the identity matrix has no affect

Some special matrices

Diagonal matrices

- Diagonal matrices are similar to $\mathbf{I}$. All non-diagonal entries are 0s, but non-zero entries can only be in the main diagonal
- Example:

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Some special matrices

Upper/lower triangular matrices

- Triangular matrices are useful in many applications
- An upper triangular matrix have all 0s below main diagonal. Example:

$$\begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- An lower triangular matrix have all 0s above main diagonal. Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 8 & 7 & 1 \end{bmatrix}$$

Symmetric matrices

- Symmetric matrices arise in many applications, including in ML/NLP (e.g., similarity or distance matrices)
- A symmetric matrix $\mathbf{A}$ satisfies $a_{ij} = a_{ji}$, or $\mathbf{A} = \mathbf{A}^T$
- Example:

$$\begin{bmatrix} 1 & 4 & -2 \\ 4 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$

- Symmetric matrices have some interesting properties (that we will return later)

Matrix–vector multiplication

- An $n \times m$ matrix can be multiplied with a m-vector to yield a n-vector

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- Example

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \times 0 + 1 \times 1 + 0 \times 0 \\ 1 \times 0 + 0 \times 1 + 1 \times 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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- One view of this operation: each entry in the resulting vector is a dot product (of rows of the matrix and the vector)

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- One view of this operation: each entry in the resulting vector is a dot product (of rows of the matrix and the vector)
- Another: the result is a linear combination of the columns of the matrix (with the entries in the vector as coefficients)

$$0 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Matrix multiplication as linear transformation

- Multiplying a vector with a matrix transforms the vector
- The result is another vector (possibly in a different vector space)
- Many operations on vectors can be expressed with multiplying with a matrix (linear transformations)

Transformation examples

identity

- Identity transformation maps a vector to itself
- For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transformation examples

permutation

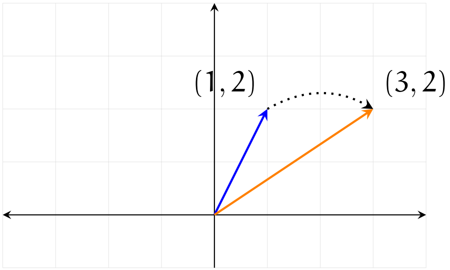
- Another simple transformation is to permute (re-arrange) the elements (rows) of the vector
- For example:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

Matrices transform vectors

stretch along the x axis

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



Transformation examples

rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Transformation examples

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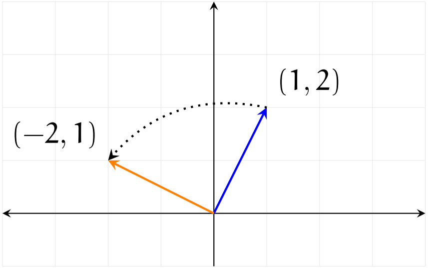

Transformations by rectangular matrices

- Multiplying a vector with (compatible) rectangular matrix results in a vector with different dimensionality
- Example $\mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Example $\mathbb{R}^3 \to \mathbb{R}^4$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Dot product as matrix multiplication

In machine learning (and many other disciplines), we treat an n-vector as an $n \times 1$ matrix.

Then, the *dot product* of two vectors is

$$\mathbf{u}^\mathsf{T}\boldsymbol{v}$$

For example, $\mathbf{u} = (2, 2)$ and $\boldsymbol{v} = (2, -2)$,

$$\begin{bmatrix} 2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

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For example, $\mathbf{u} = (2, 2)$ and $\boldsymbol{v} = (2, -2)$,

$$\begin{bmatrix} 2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \times 2 + 2 \times -2 = 4 - 4 = 0$$

- This is a 1×1 matrix, but matrices and vectors with single entries are often treated as scalars

Question: What is the transformation performed by dot product?

Outer product

The *outer product* of two column vectors is defined as

$$\boldsymbol{v}\boldsymbol{u}^{\mathsf{T}}$$

$$\begin{bmatrix}1\\2\end{bmatrix} \times \begin{bmatrix}1 & 2 & 3\end{bmatrix} =$$

Outer product

The *outer product* of two column vectors is defined as

$$\boldsymbol{v}\boldsymbol{u}^{\mathsf{T}}$$

$$\begin{bmatrix}1\\2\end{bmatrix} \times \begin{bmatrix}1 & 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\end{bmatrix}$$

Note:

- The result is a matrix
- The vectors do not have to be the same length

Matrix multiplication

- if $\mathbf{A}$ is a $n \times k$ matrix, and $\mathbf{B}$ is a $k \times m$ matrix, their product $\mathbf{C}$ is a $n \times m$ matrix
- Elements of C, $c_{i,j}$, are defined as

$$c_{ij} = \sum_{\ell=1}^{k} a_{i\ell} b_{\ell j}$$

- Note: $c_{i,j}$ is the dot product of the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$

Matrix multiplication

(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{pmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \dots a_{1k}b_{k1}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication
(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \ldots & b_{km} \end{pmatrix}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \ldots a_{1k}b_{k2}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1m} \\ c_{21} & c_{22} & \ldots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nm} \end{pmatrix}$$

Matrix multiplication

(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{pmatrix}$$

$$c_{1m} = a_{11}b_{1m} + a_{12}b_{2m} + \dots a_{1k}b_{km}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication

(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{pmatrix}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \dots a_{2k}b_{k1}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication

(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{pmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + \dots a_{2k}b_{k2}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication

(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{pmatrix}$$

$$c_{2m} = a_{21}b_{1m} + a_{22}b_{2m} + \dots a_{2k}b_{km}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication

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$$c_{n1} = a_{n1}b_{11} + a_{n2}b_{22} + \dots a_{nk}b_{k1}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication

(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{pmatrix}$$

$$c_{n2} = a_{n1}b_{12} + a_{n2}b_{22} + \dots a_{nk}b_{k2}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication

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$$c_{nm} = a_{n1}b_{1m} + a_{n2}b_{2m} + \dots a_{nk}b_{km}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication

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$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots a_{ik}b_{kj}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication example

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} =$$

Matrix multiplication example

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$$

Complexity of matrix multiplication

- How many scalar multiplications do we perform when multiplying an $n \times m$ matrix with a $m \times k$ matrix?

Complexity of matrix multiplication

- How many scalar multiplications do we perform when multiplying an $n \times m$ matrix with a $m \times k$ matrix?
  - We do $n \times k$ dot products
  - Each dot product multiplies m-dimensional vectors
  - we have $n \times m \times k$ multiplications
- In general, the complexity of matrix multiplication is $O(n^3)$
- Faster algorithms exist ($O(n^{2.81})$ or even $O(n^{2.37})$)

Properties of matrix multiplication

- Associativity

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

- Distributivity

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

- Multiplication by identity matrix

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

- Matrix multiplication is not commutative $\mathbf{AB} \neq \mathbf{BA}$ (in general)
- Matrix multiplication and transpose

$$(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$$

Question

We have three matrices:

- $\mathbf{A}$: a 10×2 matrix
- $\mathbf{B}$: a 2×5 matrix
- $\mathbf{C}$: a 5×10 matrix

- What is the dimensionality of $\mathbf{ABC}$

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- Does it matter if we perform the multiplication as
  - $(\mathbf{AB})\mathbf{C}$, or
  - $\mathbf{A}(\mathbf{BC})$

Alternative ways to think about matrix multiplication

If we have $\mathbf{AB} = \mathbf{C}$,

- Column vectors of $\mathbf{C}$, $\mathbf{c}_j = \mathbf{A}\mathbf{b}_j$
- Row vectors of $\mathbf{C}$, $\mathbf{c}_i^T = \mathbf{a}_i^T\mathbf{B}$
- $\mathbf{C}$ is also the sum of outer product of columns of $\mathbf{A}$ and rows of $\mathbf{B}$

$$C = \sum \mathbf{a}_i \mathbf{b}_i^T$$

Matrix-vector representation of a set of linear equations

The set of linear equations

$$\begin{aligned} 2x_1 &+ x_2 &= 6 \\ x_1 &+ 4x_2 &= 17 \end{aligned}$$

can be written as:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}}_{\boldsymbol{W}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 6 \\ 17 \end{bmatrix}}_{\boldsymbol{b}}$$

One can solve the above equation using row reduction, or *Gaussian elimination* (we will not cover it today).

Inverse of a matrix

(a first introduction)

- Inverse of a square matrix $\boldsymbol{W}$, when it exists, is denoted $\boldsymbol{W}^{-1}$, and defined as

$$\boldsymbol{W}\boldsymbol{W}^{-1} = \boldsymbol{W}^{-1}\boldsymbol{W} = \mathbf{I}$$

- Can you calculate the inverse of the following matrix?

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

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- Can you calculate the inverse of the following matrix?

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Once we invert the matrix, the solution to $W\mathbf{x} = \mathbf{b}$ is simply $\mathbf{x} = W^{-1}\mathbf{b}$
- We'll learn later how to calculate the inverse systematically

Summary & next week

- Vectors, matrices
- Operations between vectors and matrices
- Dot product

Next:

- Solving systems of linear equations

Further reading

- A classic reference book in the field is Strang (2009)
- Shifrin and Adams (2011) and Farin and Hansford (2014) are textbooks with a more practical/graphical orientation.
- Cherney, Denton, and Waldron (2013) and Beezer (2014) are two textbooks that are freely available.
- For more alternatives, see
  `http://www.openculture.com/free-math-textbooks`
- You may also find the MIT video lectures on introductory linear algebra at
  `https://www.youtube.com/playlist?list=PL49CF3715CB9EF31D`

Further reading (cont.)

- Beezer, Robert A. (2014). *A First Course in Linear Algebra*. version 3.40. Congruent Press. ISBN: 9780984417551. URL: `http://linear.ups.edu/`.
- Cherney, David, Tom Denton, and Andrew Waldron (2013). *Linear algebra*. math.ucdavis.edu. URL: `https://www.math.ucdavis.edu/~linear/`.
- Farin, Gerald E. and Dianne Hansford (2014). *Practical linear algebra: a geometry toolbox*. Third edition. CRC Press. ISBN: 978-1-4665-7958-3.
- Shifrin, Theodore and Malcolm R Adams (2011). *Linear Algebra. A Geometric Approach*. 2nd. W. H. Freeman. ISBN: 978-1-4292-1521-3.
- Strang, Gilbert (2009). *Introduction to Linear Algebra, Fourth Edition*. 4th ed. Wellesley Cambridge Press. ISBN: 9780980232714.